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Bogomolny Field Equations and the Double-Complex Function Method

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The double-complex function method is applied to the Bogomolny field equations of Yang – Mills – Higgs theory. We discuss how to generate a family of solutions of the Bogomolny field equations with a double-complex Ernst potential. We discuss the application of the Neugebauer – Kramer transformation, the double Ehlers transformation, and noncommutativity. An example of a concrete calculation is given.

1. INTRODUCTION AND PRELIMINARIES

Manton (1978) and Forgacs *et al.* (1980) pointed out that in Yang–Mills–Higgs theory, if we give the $SU(2)$ gauge field W^a and the Higgs field Φ^a the axisymetric ansatz

$$\begin{aligned} \Phi^a &= (0, \varphi_1, \varphi_2), \qquad & W^a_\varphi &= -(0, \eta_1, \eta_2) \\ W^a_z &= (W_1, 0, 0), \qquad & W^a_\rho &= -(W_2, 0, 0) \end{aligned} \tag{1}$$

where φ, ρ, z are the usual polar coordinates and φ_i, W_i, η_i are functions of ρ, z only, then the Bogomolny field equations become

$$\begin{aligned} \sqrt{A^2 - C^2}(\partial_\rho\varphi_1 - W_2\varphi_2) &= -(\partial_z\eta_1 - W_1\eta_2)/\rho \\ \sqrt{A^2 - C^2}(\partial_\rho\varphi_2 + W_2\varphi_1) &= -(\partial_z\eta_2 + W_1\eta_1)/\rho \\ \sqrt{A^2 - C^2}(\varphi_1\eta_2 - \varphi_2\eta_1) &= \rho(\partial_\rho W_1 - \partial_z W_2) \\ \sqrt{A^2 - C^2}(\partial_z\varphi_1 - W_1\varphi_2) &= (\partial_\rho\eta_1 - W_2\eta_2)/\rho \\ \sqrt{A^2 - C^2}(\partial_z\varphi_2 + W_1\varphi_1) &= (\partial_\rho\eta_2 + W_2\eta_1)/\rho \end{aligned} \tag{2}$$

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Further, Forgacs *et al.* (1981) pointed out that if φ_i, η_i, and W_i are written as

$$\begin{aligned} \sqrt{A^2 - C^2}\varphi_1 &= -W_1 = \frac{1}{f}\partial_z\psi, & \eta_1 &= \rho W_2 = -\frac{\rho}{f}\partial_\rho\psi \\ \sqrt{A^2 - C^2}\varphi_1 &= -\frac{1}{f}\partial_z f, & \eta_2 &= \frac{\rho}{f}\partial_\rho f \end{aligned} \tag{3}$$

then the Bogomolny field equations (2) are equivalent to the Ernst equation of general relativity (Ernst, 1968)

$$\begin{aligned} &\mathrm{Re}(\mathscr{E})\nabla^2\mathscr{E} = \nabla\mathscr{E} \cdot \nabla\mathscr{E} \\ &\nabla^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2, \qquad \nabla = (\partial_\rho, \partial_z) \\ &\mathscr{E} = f + i\psi \end{aligned} \tag{4}$$

Recently, by the use of the above results, Singleton (1996) discussed the Kerr-like solution for the Bogomolny fields and its physical behavior.

Zhong (1985, 1988, 1989, 1990a,b) extended the ordinary Ernst equation to a double-complex form, and applied it to general relativity, soliton theory, and self-dual $SU(2)$ gauge fields. Here we apply the double-complex method to the Bogomolny fields of Yang–Mills–Higgs theory. We discuss how to generate the solutions for the Bogomolny field equations with the double-complex method so the number of solutions for the Bogomolny field equations is increased greatly. In particular, we can still use the double Ehlers transformation to get new solutions for the Bogomolny field equations, and the method reflects a symmetry structure of the Bogomolny fields.

For convenience, we collect here some results concerning the double-complex function method. Let J denote the double pure imaginary unit, i.e., $J = i$ ($i^2 = -1$) or $J = \varepsilon$ ($\varepsilon^2 = 1$, $\varepsilon \neq \pm 1$). If the a_n are real numbers and the real series $\Sigma\, a_n$ is absolutely convergent, then

$$a(J) = \sum_{n=0}^{\infty} a_n J^{2n} \tag{5}$$

is called a double-real number. If $a(J)$ and $b(J)$ are both double-real numbers, $Z(J) = a(J) + J \cdot b(J)$ is called a double-complex number. Sometimes $Z(J)$ is written directly as a pair of dual complex numbers, i.e., $Z(J) = (Z_C, Z_H)$, where $Z_C = Z(J = i)$, $Z_H = Z(J = \varepsilon)$.

The double-complex Ernst equation is (Zhong, 1985)

$$\mathrm{Re}(\mathscr{E}(J))\nabla^2\mathscr{E}(J) = \nabla\mathscr{E}(J) \cdot \nabla\mathscr{E}(J), \qquad \mathscr{E}(J) = F(J) + J \cdot \Omega(J) \tag{6}$$

The superiority of the double-complex Ernst equation lies in the fact that when we find a solution $\mathscr{E}(J)$, we acquire an ordinary-complex solution and a hyperbolic-complex solution $(\mathscr{E}_C, \mathscr{E}_H) = (F_C + i\Omega_C, F_H + \varepsilon\Omega_H)$. For work on this equation see, e.g., Zhong (1985, 1988, 1989, 1990a,b), Gao and Zhong (1991, 1996), and Feng and Zhong (1996).

This paper is organized as follows: In Section 2 we examine how to get four solutions for the Bogomolny fields by a double-complex Ernst potential, and calculate an example. In Section 3 we discuss how to obtain new families of solutions for the Bogomolny fields by the use of the double Ehlers transformation and noncommutativity. In Section 4 we apply the preceding method to a particular case, and use Weyl's solution of general relativity to generate new solutions for the Bogomolny fields. Section 5 contains a conclusion and discussion.

2. GENERATING THE SOLUTIONS OF THE BOGOMOLNY FIELD EQUATIONS FROM A DOUBLE-COMPLEX ERNST POTENTIAL

Suppose that a double-complex Ernst potential $\mathscr{E}(J)$ obeying Eq. (6) is given. First, applying the ordinary-complex Ernst potential $\mathscr{E}_C = F_C + i\Omega_C$ to Eq. (4), we can get a Bogomolny field solution (3). Then, for the hyperbolic complex Ernst potential $\mathscr{E}_H = F_H + \varepsilon\Omega_H$, we can turn it into the ordinary Ernst potential $\mathscr{D}_C$ which is dual to $\mathscr{E}_H$ by way of the Neugebauer–Kramer (1969) transformation as follows: We use the notation in Zhong (1989) to denote the transformation $d_H = (T, W_{F_H})$:

$$
\begin{aligned}
&d_H: \quad \mathscr{E}_H = F_H + \varepsilon\Omega_H \to \mathscr{D}_C = G_C + i\Theta_C \\
&T: \quad F_H \to G_C = T(F_H) = \rho/F_H \\
&W_{F_H}: \quad \Omega_H \to \Theta_C = \int \frac{\rho}{F_H^2}(\partial_z\Omega_H d\rho - \partial_\rho\Omega_H dz)
\end{aligned}
\tag{7}
$$

i.e.,

$$
\partial_\rho\Theta_C = \frac{\rho}{F_H^2}\,\partial_z\Omega_H, \qquad \partial_z\Theta_C = -\frac{\rho}{F_H^2}\,\partial_\rho\Omega_H
$$

Now, from the $\mathscr{D}_C$ we obtain another solution (3). In addition, we note that in general relativity, the two stationary axisymmetric gravitational field solutions obtained from a double-complex Ernst potential $\mathscr{E}(J)$ and its conjugate double-complex Ernst potential $\mathscr{E}^*(J)$ are actually equivalent. For the Bogomolny fields, however, this is not so; in fact, the two Bogomolny solutions corresponding respectively to $\mathscr{E}(J)$ and $\mathscr{E}^*(J)$ are different. Thus we can get four solutions for the Bogomolny fields from a known $\mathscr{E}(J)$. In order to write out clearly the four solutions, we define a transformation $\mathscr{L}_C$

that changes the ordinary-complex Ernst potential $\mathscr{E}_C$ into a Bogomolny solution (φ_i, W_i, η_i), according to Eq. (3), as follows:

$$\mathscr{L}_C\colon \quad \mathscr{E}_C \to (\varphi_i, W_i, \eta_i) = \mathscr{L}_C(\mathscr{E}_C)$$

$$\sqrt{A^2 - C^2}\varphi_1 = -W_1 = \frac{1}{F_C}\partial_z\Omega_C, \qquad \eta_1 = \rho W_2 = -\frac{\rho}{F_C}\partial_\rho\Omega_C \tag{8}$$

$$\sqrt{A^2 - C^2}\varphi_2 = -\frac{1}{F_C}\partial_z F_C, \qquad \eta_2 = \frac{\rho}{F_C}\partial_\rho F_C$$

Furthermore, we define another transformation $\mathscr{L}_H = \mathscr{L}_C \cdot d_H$ which turns a hyperbolic-complex Ernst potential $\mathscr{E}_H$ into another Bogomolny solution $(\tilde{\varphi}_i, \tilde{W}_i, \tilde{\eta}_i)$. Putting (7) into (8), we get the other Bogomolny solution for $\mathscr{E}_H$,

$$\mathscr{L}_H\colon \quad \mathscr{E}_H \to (\tilde{\varphi}_i, \tilde{W}_i, \tilde{\eta}_i) = \mathscr{L}_H(\mathscr{E}_H)$$

$$\sqrt{A^2 - C^2}\tilde{\varphi}_1 = -\tilde{W}_1 = -\frac{1}{F_H}\partial_\rho\Omega_H, \qquad \tilde{\eta}_1 = \rho\tilde{W}_2 = -\frac{\rho}{F_H}\partial_z\Omega_H \tag{9}$$

$$\sqrt{A^2 - C^2}\tilde{\varphi}_2 = \frac{1}{F_H}\partial_z F_H, \qquad \tilde{\eta}_2 = 1 - \frac{\rho}{F_H}\partial_\rho F_H$$

As a example, let us take (Zhong, 1985)

$$\mathscr{E}(J) = \frac{\xi(J) - 1}{\xi(J) + 1}, \qquad \xi(J) = e^{J\alpha}\Psi, \qquad \Psi = -\coth\theta = \frac{e^{-\theta} + e^{\theta}}{e^{-\theta} - e^{\theta}} \tag{10}$$

where $\theta = \theta(\rho, z)$ is a harmonic function, i.e., $\nabla^2\theta = 0$; α is a real constant; and $e^{J\alpha} = C[J;\alpha] + J \cdot S[J;\alpha]$ is a double-exponential function, where the double-cosine and the double-sine are defined, respectively, as

$$\begin{aligned} C[J;\alpha] &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} J^{2n}\alpha^{2n} = \begin{cases} \cos\alpha, & J = i \\ \cosh\alpha, & J = \varepsilon \end{cases} \\ S[J;\alpha] &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} J^{2n}\alpha^{2n+1} = \begin{cases} \sin\alpha, & J = i \\ \sinh\alpha, & J = \varepsilon \end{cases} \end{aligned} \tag{11}$$

Then

$$\mathscr{E}(J) = \frac{e^{J\alpha}\Psi - 1}{e^{J\alpha}\Psi + 1} = \frac{(\Psi^2 - 1) + J \cdot 2(S[J;\alpha]\Psi)}{\Psi^2 + 2C[J;\alpha]\Psi + 1}$$

i.e.,

$$F(J) = \frac{\Psi^2 - 1}{\Psi^2 + 2C[J;\alpha] + 1}, \qquad \Omega(J) = \frac{2S[J;\alpha]\Psi}{\Psi^2 + 2C[J;\alpha]\Psi + 1} \tag{12}$$

The four solutions of the Bogomolny field equations are as follows: From

$$\mathscr{E}_C = F_C + i\Omega_C = \frac{\Psi^2 - 1}{\Psi^2 + 2\cos\alpha\Psi + 1} + i\,\frac{2\sin\alpha\Psi}{\Psi^2 + 2\cos\alpha\Psi + 1}$$

we get

$$\begin{aligned}
\sqrt{A^2 - C^2}\varphi_1 &= -W_1 = \frac{1}{F_C}\,\partial_z\Omega_C = \frac{2\sin\alpha\,\mathrm{csch}^2\theta}{\Psi^2 + 2\cos\alpha\Psi + 1}\,\partial_z\theta \\
\eta_1 &= \rho W_2 = -\frac{\rho}{F_C}\,\partial_\rho\Omega_C = -\frac{2\rho\sin\alpha\,\mathrm{csch}^2\theta}{\Psi^2 + 2\cos\alpha\Psi + 1}\,\partial_\rho\theta \\
\sqrt{A^2 - C^2}\varphi_2 &= -\frac{1}{F_C}\,\partial_z F_C = \frac{(2\cos\alpha\Psi^2 + 2\Psi + \cos\alpha)\,\mathrm{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cos\alpha\Psi + 1)}\,\partial_z\theta \\
\eta_2 &= \frac{\rho}{F_C}\,\partial_\rho F_C = -\frac{2\rho(\cos\alpha\Psi^2 + 2\Psi + \cos\alpha)\,\mathrm{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cos\alpha\Psi + 1)}\,\partial_\rho\theta
\end{aligned} \tag{13}$$

From $\mathscr{E}_C^* = F_C - i\Omega_C$, we get

$$\begin{aligned}
\sqrt{A^2 - C^2}\varphi_1 &= -W_1 = -\frac{2\sin\alpha\,\mathrm{csch}^2\theta}{(\Psi^2 + 2\cos\alpha\Psi + 1)}\,\partial_z\theta \\
\eta_1 &= \rho W_2 = \frac{2\sin\alpha\,\mathrm{csch}^2\theta}{\Psi^2 + 2\cos\alpha\Psi + 1}\,\partial_\rho\theta \\
\sqrt{A^2 - C^2}\varphi_2 &= -\frac{1}{F_C}\,\partial_z F_C = \frac{(2\cos\alpha\Psi^2 + 2\Psi + \cos\alpha)\,\mathrm{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cos\alpha\Psi + 1)}\,\partial_z\theta \\
\eta_2 &= \frac{\rho}{F_C}\,\partial_\rho F_C = -\frac{2\rho(\cos\alpha\Psi^2 + 2\Psi + \cos\alpha)\,\mathrm{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cos\alpha\Psi + 1)}\,\partial_\rho\theta
\end{aligned} \tag{14}$$

From

$$\mathscr{E}_H = F_H + \varepsilon\Omega_H = \frac{\Psi^2 - 1}{\Psi^2 + 2\cosh\alpha\Psi + 1} + \varepsilon\,\frac{2\sinh\alpha\Psi}{\Psi^2 + 2\cosh\alpha\Psi + 1}$$

we get

$$\sqrt{A^2 - C^2}\varphi_1 = -W_1 = -\frac{1}{F_H}\partial_\rho \Omega_H = -\frac{2 \sin h\alpha \operatorname{csch}^2\theta}{\Psi^2 + 2\cosh\alpha\Psi + 1}\partial_\rho\theta$$

$$\eta_1 = \rho W_2 = -\frac{\rho}{F_H}\partial_z \Omega_H = -\frac{2\rho \sinh\alpha \operatorname{csch}^2\theta}{\Psi^2 + 2\cosh\alpha\Psi + 1}\partial_z\theta$$

$$\sqrt{A^2 - C^2}\varphi_1 = \frac{1}{F_H}\partial_z F_H = -\frac{(2\cosh\alpha\Psi^2 + 2\Psi + \cosh\alpha)\operatorname{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cosh\alpha\Psi + 1)}\partial_z\theta$$

$$\eta_2 = 1 - \frac{\rho}{F_H}\partial_\rho F_H = 1 + \frac{(2\cosh\alpha\Psi^2 + 2\Psi + \cosh\alpha)\operatorname{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cosh\alpha\Psi + 1)}\partial_\rho\theta \tag{15}$$

From $\mathscr{E}_H^* = F_H - \varepsilon\Omega_H$, we get

$$\sqrt{A^2 - C^2}\varphi_1 = -W_1 = \frac{1}{F_H}\partial_\rho \Omega_H = \frac{2\sinh\alpha \operatorname{csch}^2\theta}{\Psi^2 + 2\cosh\alpha\Psi + 1}\partial_\rho\theta$$

$$\eta_1 = \rho W_2 = \frac{\rho}{F_H}\partial_z \Psi_H = \frac{2\rho \sinh\alpha \operatorname{csch}^2\theta}{\Psi^2 + 2\cosh\alpha\Psi + 1}\partial_z\theta \tag{16}$$

$$\sqrt{A^2 - C^2}\varphi_2 = -\frac{1}{F_H}\partial_z F_H = -\frac{(2\cosh\alpha\Psi^2 + 2\Psi + \cosh\alpha)\operatorname{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cosh\alpha\Psi + 1)}\partial_z\theta$$

$$\eta_2 = 1 - \frac{\rho}{F_H}\partial_\rho F_H = 1 + \frac{(2\cosh\alpha\Psi^2 + 2\Psi + \cosh\alpha)\operatorname{csch}^2\theta}{(\Psi^2 - 1)(\Psi^2 + 2\cosh\alpha\Psi + 1)}\partial_\rho\theta$$

3. GENERATING NEW SOLUTIONS FOR THE BOGOMOLNY FIELD BY THE DOUBLE-EHLERS TRANSFORMATION

For a known double-complex Ernst potential, Zhong (1985) gave a double Ehlers transformation $u(J)$,

$$u(J): \quad \mathscr{E}(J) \to \tilde{\mathscr{E}}(J) = \frac{a(J)\mathscr{E}(J) + Jb(J)}{Jc(J)\mathscr{E}(J) + d(J)} \tag{17}$$

where $a(J)$, $b(J)$, $c(J)$, and $d(J)$ are double-real constants, satisfy the condition $a(J)b(J)\text{-}J^2b(J)c(J) = 1$. Then $\tilde{\mathscr{E}}(J)$ still satisfies Eq. (6). Consider the transformations $u_C = u(J = i)$ and $u_H = u(J = \varepsilon)$ and the dual mapping

$d_C = (T, V_{F_C})$ which changes an ordinary-complex Ernst potential $\mathscr{E}_C$ into a hyperbolic-complex Ernst potential $\mathscr{D}_H$,

$$
\begin{aligned}
&d_C: \quad \mathscr{E}_C \to \mathscr{D}_H = d_C(\mathscr{E}_C) = G_H + \varepsilon\Theta_H \\
&T: \quad F_C \to G_H = T(F_C) = \frac{\rho}{F_C} \\
&V_{F_C}: \quad \Omega_C \to \Theta_H = V_{F_C}(\Omega_C) = \int \frac{\rho}{F_C{}^2}(-\partial_z\Omega_C d\rho + \partial_\rho\Omega_C dz)
\end{aligned}
\tag{18}
$$

i.e.,

$$
\partial_\rho\Theta_H = -\frac{\rho}{F_C^2}\partial_z\Omega_C, \qquad \partial_z\Theta_H = \frac{\rho}{F_C^2}\partial_\rho\Omega_C
$$

Note that d_C is not the inverse of d_H. In fact, it is easy to check that $d_C \cdot d_H = * = d_H \cdot d_C$, where $*$ is the complex conjugate operation.

Making use of u_C, u_H, d_C, d_H, $\mathscr{L}_C$, and $\mathscr{L}_H$, we can draw the following diagram of the generating of Bogomolny field solutions from a double-complex Ernst potential $\mathscr{E}(J) = F(J) + J\Omega(J)$:

$$
\begin{array}{ccccc}
(G_H, \Theta_H) & \xrightarrow{u_H} & (\tilde{G}_H, \tilde{\Theta}_H) & \xrightarrow{\mathscr{L}_H = \mathscr{L}_C \cdot d_H} & (\tilde{\varphi}'_i, \tilde{W}'_i, \tilde{\eta}'_i)_{(H)} \\
\uparrow d_C & & \downarrow d_H & & \\
(F_C, \Omega_C) & \xrightarrow{u_C} & (\tilde{F}_C, \tilde{\Omega}_C) & \xrightarrow{\mathscr{L}_C} & (\tilde{\varphi}_i, \tilde{W}_i, \tilde{\eta}_i)_{(C)} \\
 & & & & \\
(F_H, \Omega_H) & \xrightarrow{u_H} & (\tilde{F}_H, \tilde{\Omega}_H) & \xrightarrow{\mathscr{L}_H = \mathscr{L}_C \cdot d_H} & (\tilde{\varphi}_i, \tilde{W}_i, \tilde{\eta}_i)_{(H)} \\
\downarrow d_H & & \uparrow d_C & & \\
(G_C, \Theta_C) & \xrightarrow{u_C} & (\tilde{G}_C, \tilde{\Theta}_C) & \xrightarrow{\mathscr{L}_C} & (\tilde{\varphi}'_i, \tilde{W}'_i, \tilde{\eta}'_i)_{(C)}
\end{array}
\tag{19}
$$

It is easy to verify the noncommutativity of the following diagram:

$$
\begin{array}{ccc}
\mathscr{E}_C & \xrightarrow{u_C} & \tilde{\mathscr{E}}_C \neq \tilde{\mathscr{E}}'_C \\
\downarrow d_C & & \uparrow d_H \\
\mathscr{D}_H & \xrightarrow{u_H} & \tilde{\mathscr{D}}_H
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{E}_H & \xrightarrow{u_H} & \tilde{\mathscr{E}}_H \neq \tilde{\mathscr{E}}'_H \\
\downarrow d_H & & \uparrow d_C \\
\mathscr{D}_C & \xrightarrow{u_C} & \tilde{\mathscr{D}}_C
\end{array}
\tag{20}
$$

$$
u_C \neq d_H \cdot u_H \cdot d_C \qquad\qquad u_H \neq d_C \cdot u_C \cdot d_H
$$

Thus, by noncommutativity and the results in Section 2, from $\mathscr{E}(J)$ we can get the following solution chains of Bogomolny equations:

$$
\begin{array}{l}
(\varphi_i', W_i', \eta_i')_{(C)} \xleftarrow{\mathscr{L}_C} (F_C, -\Omega_C) \begin{array}{l} \xrightarrow{d_H \cdot u_H \cdot d_C} (\overline{F}_C', -\overline{\Omega}_C') \xrightarrow{\mathscr{L}_C} (\overline{\varphi}_i', \overline{W}_i', \overline{\eta}_i')_{(C)} \\ \xrightarrow{u_C} (\overline{F}_C, -\overline{\Omega}_C) \xrightarrow{\mathscr{L}_C} (\overline{\varphi}_i, \overline{W}_i, \overline{\eta}_i)_{(C)} \end{array} \\[2ex]
(\varphi_i, W_i, \eta_i)_{(C)} \xleftarrow{\mathscr{L}_C} (F_C, \Omega_C) \begin{array}{l} \xrightarrow{d_H \cdot u_H \cdot d_C} (\tilde{F}_C', \tilde{\Omega}_C') \xrightarrow{\mathscr{L}_C} (\tilde{\varphi}_i', \tilde{W}_i', \tilde{\eta}_i')_{(C)} \\ \xrightarrow{u_C} (\tilde{F}_C, \tilde{\Omega}_C) \xrightarrow{\mathscr{L}_C} (\tilde{\varphi}_i, \tilde{W}_i, \tilde{\eta}_i)_{(C)} \end{array} \\[2ex]
(\varphi_i, W_i, \eta_i)_{(H)} \xleftarrow{\mathscr{L}_H} (F_H, \Omega_H) \begin{array}{l} \xrightarrow{u_H} (\tilde{F}_H', \tilde{\Omega}_H') \xrightarrow{\mathscr{L}_H} (\tilde{\varphi}_i', \tilde{W}_i', \tilde{\eta}_i')_{(H)} \\ \xrightarrow{d_C \cdot u_C \cdot d_H} (\tilde{F}_H, \tilde{\Omega}_H) \xrightarrow{\mathscr{L}_H} (\tilde{\varphi}_i, \tilde{W}_i, \tilde{\eta}_i)_{(H)} \end{array} \\[2ex]
(\varphi_i', W_i', \eta_i')_{(H)} \xleftarrow{\mathscr{L}_H} (F_H, -\Omega_H) \begin{array}{l} \xrightarrow{u_H} (\overline{F}_H', -\overline{\Omega}_H') \xrightarrow{\mathscr{L}_H} (\overline{\varphi}_i', \overline{W}_i', \overline{\eta}_i')_{(H)} \\ \xrightarrow{d_C \cdot u_C \cdot d_H} (\overline{F}_H, -\overline{\Omega}_H) \xrightarrow{\mathscr{L}_H} (\overline{\varphi}_i, \overline{W}_i, \overline{\eta}_i)_{(H)} \end{array}
\end{array}
\tag{21}
$$

In the above chain every step may be carried out continuously. For example,

$$
(F_C, \Omega_C) \begin{array}{l} \xrightarrow{u_C} (F_{C1}^{(1)}, \Omega_{C1}^{(1)}) \begin{array}{l} \nearrow (F_{C1}^{(2)}, \Omega_{C1}^{(2)}) \begin{array}{l} \nearrow \cdots \\ \searrow \cdots \end{array} \\ \searrow (F_{C2}^{(2)}, \Omega_{C2}^{(2)}) \begin{array}{l} \nearrow \cdots \\ \searrow \cdots \end{array} \end{array} \\ \xrightarrow{d_H \cdot u_H \cdot d_C} (F_{C2}^{(1)}, \Omega_{C2}^{(1)}) \begin{array}{l} \nearrow (F_{C3}^{(2)}, \Omega_{C3}^{(2)}) \begin{array}{l} \nearrow \cdots \\ \searrow \cdots \end{array} \\ \searrow (F_{C4}^{(2)}, \Omega_{C4}^{(2)}) \begin{array}{l} \nearrow \cdots \\ \searrow \cdots \end{array} \end{array} \end{array}
\tag{22}
$$

Obviously, for an $\mathscr{E}(J)$, after n steps, we obtain $2^2 + 2^3 + \cdots + 2^{n+2} = 4(2^{n+1} - 1)$ Bogomolny field solutions.

4. THE SOLUTION FAMILIES OF THE BOGOMOLNY FIELDS GENERATED BY THE WEYL SOLUTION

In general relativity there is a simple stationary axisymmetric solution, the Weyl solution, which is a real solution $\mathscr{E} = e^h$, where $h(\rho, z)$ is a harmonic function, $\nabla^2 h = 0$. In this case $\mathscr{E} = \mathscr{E}_C = \mathscr{E}_H = e^h$. According to (8), the solutions for the Bogomolny field equations corresponding to $\mathscr{E}_C = \mathscr{E}_C^*$ are

$$\begin{aligned} \sqrt{A^2 - C^2}\varphi_1 &= -W_1 = 0, \qquad & \eta_1 &= \rho W_2 = 0 \\ \sqrt{A^2 - C^2}\varphi_2 &= -\partial_z h, & \eta_2 &= \rho\partial_\rho h \end{aligned} \tag{23}$$

Similarly, according to Eq. (9), the solutions corresponding to $\mathscr{E}_H = \mathscr{E}_H^* = e^h$ are

$$\begin{aligned} \sqrt{A^2 - C^2}\varphi_1 &= -W_1 = 0, \qquad & \eta_1 &= \rho W_2 = 0 \\ \sqrt{A^2 - C^2}\varphi_2 &= \partial_z h, & \eta_2 &= 1 - \rho\partial_\rho h \end{aligned} \tag{24}$$

Now, let the double Ehlers transformation act on $\mathscr{E}(J)$,

$$\begin{aligned} u(J): \quad \mathscr{E} \to \tilde{\mathscr{E}}(J) &= \frac{a(J)\mathscr{E}(J) + Jb(J)}{Jc(J)\mathscr{E} + d(J)} = \frac{e^h + J[b(J)d(J) - a(J)c(J)e^{2h}]}{d^2(J) - J^2c^2(J)e^{2h}} \\ &= \tilde{F}(J) + J\Omega(J) \\ a(J)d(J) &- J^2b(J)c(J) = 1 \end{aligned} \tag{25}$$

From this we obtain

$$\begin{aligned} \tilde{F}_C &= \frac{e^h}{d_C^2 + c_C^2 e^{2h}}, \qquad & \tilde{F}_H &= \frac{e^h}{d_H^2 - c_H^2 e^{2h}} \\ \Omega_C &= \frac{b_C d_C - a_C c_C e^{2h}}{d_C^2 + c_C^2 e^{2h}}, & \Omega_H &= \frac{b_H d_H - a_H c_H e^{2h}}{d_H^2 - c_H^2 e^{2h}} \end{aligned} \tag{26}$$

The Bogomolny field solution obtained from $\tilde{\mathscr{E}}_C = \tilde{F}_C + i\Omega_C$ is

$$\begin{aligned} \sqrt{A^2 - C^2}\varphi_1 &= -W_1 = \frac{-2c_C d_C e^h}{d_C^2 + c_C^2 e^{2h}}\partial_z h \\ \eta_1 &= \rho W_2 = \frac{-2c_C d_C \rho e^h}{d_C^2 + c_C^2 e^{2h}}\partial_\rho h \\ \sqrt{A^2 - C^2}\varphi_2 &= \frac{c_C^2 e^{2h} - d_C^2}{d_C^2 + c_C^2 e^{2h}}\partial_z h \\ \eta_2 &= \frac{\rho(d_C^2 - c_C^2 e^{2h})}{d_C^2 + c_C^2 e^{2h}}\partial_\rho h \end{aligned} \tag{27}$$

The Bogomolny solution obtained from $\tilde{\mathscr{E}}_H = \tilde{F}_H + \varepsilon\Omega_H$ is

$$\sqrt{A^2 - C^2}\varphi_1 = -W_1 = \frac{2c_H d_H e^h}{d_H^2 - c_H^2 e^{2h}}\partial_\rho h$$

$$\eta_1 = \rho W_2 = \frac{2c_H d_H e^h \rho}{d_H^2 - c_H^2 e^{2h}}\partial_z h \tag{28}$$

$$\sqrt{A^2 - C^2}\varphi_2 = \frac{d_H^2 + c_H^2 e^{2h}}{d_H^2 - c_H^2 e^{2h}}\partial_z h$$

$$\eta_2 = 1 - \frac{d_H^2 + c_H^2 e^{2h}}{d_H^2 - c_H^2 e^{2h}}\partial_\rho h$$

Now, let another transformation $d_C \cdot u_C \cdot d_H$ act on $\mathscr{E}_H$. First, let d_H act on $\mathscr{E}_H$,

$$d_H: \quad \mathscr{E}_H \to \mathscr{D}_C = G_C + i\Theta_C = \frac{\rho}{e^h} \tag{29}$$

Then let u_C act on $\mathscr{D}_C$,

$$u_C: \quad \mathscr{D}_C \to \tilde{\mathscr{D}}_C = \tilde{G}_C + i\Theta_C = \frac{\rho e^h}{d_C^2 e^{2h} + c_C^2\rho^2} + i\,\frac{b_C d_C e^{2h} - a_C c_C \rho^2}{d_C^2 e^{2h} + c_C^2 \rho^2} \tag{30}$$

Finally, let d_C act on $\tilde{\mathscr{D}}_C$; then we obtain $d_C \cdot u_C \cdot d_H(\mathscr{E}_H)$,

$$\begin{aligned} d_C: \quad \tilde{\mathscr{D}} &\to \tilde{\mathscr{E}}'_H \\ &= \tilde{F}'_H + \varepsilon\Omega_{H'} \\ &= \frac{d_C^2 \mathrm{e}^{2h} + c_C^2\rho^2}{e^h} \\ &\quad + \varepsilon \int [-2c_C d_C \rho\partial_z h d\rho + 2c_C d_C(\rho\partial_\rho h - 1)\, dz] \end{aligned} \tag{31}$$

The Bogomolny field solution obtained from $\tilde{\mathscr{E}}'_H = d_C \cdot u_C \cdot d_H(\mathscr{E}_H)$ is

$$\sqrt{A^2 - C^2}\varphi_1 = -W_1 = \frac{2c_C d_C \rho e^h\, \partial_z h}{d_C^2 e^{2h} + c_C^2 \rho^2}$$

$$\eta_1 = \rho W_2 = \frac{2c_C d_C \rho e^h (1 - \rho\partial_\rho h)}{d_C^2 e^{2h} + c_C^2\rho^2} \tag{32}$$

$$\sqrt{A^2 - C^2}\varphi_2 = \frac{d_C^2 e^{2h} - c_C^2\rho^2}{d_C^2 e^{2h} + c_C^2\rho^2}\partial_z h$$

$$\eta_2 = 1 - \frac{2c_C^2\rho^2 + \rho(d_C^2 e^{2h} - c_C^2\rho^2)}{d_C^2 e^{2h} + c_C^2\rho^2}\,\partial_\rho h$$

Here we can see indeed that the Bogomolny field solutions obtained from $\tilde{\mathscr{E}}_H = u_H(\mathscr{E}_H)$ and $\tilde{\mathscr{E}}'_H = d_C \cdot u_C \cdot d_H(\mathscr{E}_H)$ are different.

We have calculated the Bogomolny field solution family corresponding to $\mathscr{E}_C$, $\mathscr{E}_C^*$, $\mathscr{E}_H$, $\mathscr{E}_C^*$, $u_C(\mathscr{E}_C)$, $u_H(\mathscr{E}_H)$, and $d_C \cdot u_C \cdot d_H(\mathscr{E}_H)$. The calculations are similar for the other solutions in (21).

5. CONCLUSION AND DISCUSSION

The double-complex function method can be applied to the Bogomolny field equations in Yang–Mills–Higgs theory. We can obtain four Bogomolny field solutions from a known double-complex Ernst potential. In particular, by a double Ehlers transformation combining noncommutativity again and again, an infinite chain of Bogomolny field solutions can be generated. This reflects that the Bogomolny fields have a dual symmetry corresponding to the double Ehlers transformation group.

Using the method given in this paper, many results concerning the double-complex Ernst equation can be applied to the discussion of Bogomolny fields. The physical behavior of the Bogomolny field solutions generated from the double-complex Ernst potential will be discussed elsewhere.

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